A Text Book on

Industrial Engineering, Robotics and Mechatronics

Useful for IAS / GATE / ESE / PSUs and other competitive examinations

Dr. Swadesh Kumar Singh
B.Tech. AMU Aligarh, Gold Medalist
M. Tech. and Ph.D. from IIT Delhi
Ex. IES officer, Govt. of India
Professor, GRIET, Hyderabad

Dr. Rajesh Purohit
M. Tech. and Ph.D. from IIT Delhi
Associate Professor, MANIT, Bhopal

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A text book on Industrial Engineering, Robotics and Mechatronics

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First Edition: 2016

Reprint: 2019

Reprint: 2022

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FOREWORD

In my long teaching career I have come across many other fellow teachers but Dr. Swadesh Singh is unique. He has the rare combination of being Ex-IES officer in Govt. of India, a Ph.D from reputed Indian Institute of Technology Delhi, a Young Scientist award winner and a Career Award winner for teaching. As a teacher, he is one of those who really tries to understand the students from their perspective and helps them. He has a penchant for guiding and counseling students in choosing their careers and preparing them for their competitive exams. Many students have benefited from him and holds regard for him even years after graduating. Dr. Singh's book on Production Technology has become popular and is a standard among the students who are top rankers in competitive exams like GATE, CIVIL SERVICES etc. and the present book on Industrial engineering should stand testimony to the expertise of Dr. Singh's in presenting in a compact book what is just appropriate for preparing for such exams.

This book is a must-have for those who are serious in finding success in competitive exams.

Prof. P. S. Raju
Director
GRIET, Hyderabad, India

PREFACE

After the overwhelming response of my first book on Production Engineering, I thought of writing another book on the core subject of Industrial Engineering. After teaching for various competitive examinations for about 16 years, I found that although there are so many books available on the fundamentals of Industrial Engineering but there is no book available for competitive examinations. The book which student can have and crack any question that appears in examination like GATE, IES and other public sector examination. So in the present book my effort is to teach fundamentals by problem solving so that students can understand well. Apart from the solved example-problems, numerous objective-type practice problems from various competitive examinations are given, along with the answer keys. The first edition of this book I released in 2011. Since most of the work including the typesetting, editing and so on was carried out by me, there were many grammatical mistakes in that edition. While teaching and students solving the problem we have corrected the text thoroughly and I have also added some more relevant text in each topic making it more suitable for the competitive examinations. Along with Dr. Rajesh Purohit of MANIT Bhopal, I have added Robotics and Mechatronics to make the book suitable for IES and other competitive examinations.

I express my gratitude to my spiritual teacher who has given me inspiration to write this book. Every subject matter can be learned through the medium of a teacher only. He thought me both his precept and by his personal example how to be sincere at work and what is good for me and how can I help others in a genuine way.

The authors are thankful to Sunil Kumar, Deepen Banoriya, Utkarsh Pandey, Bhrant Kumar, Harish, Limbadri, Prudvi, Gangadhar and Srinivasu for taking pain in correcting the manuscript many times. We also thank MADE EASY proof editing staff and CMD Sri B. Singh for their full support in publishing this book.

We hope that aspirants of scoring high in GATE, ESE, IAS, PSU and any competitive exams find this book as a helpful aid in their preparations.

Dr. Swadesh Kumar Singh

Dr. Rajesh Purohit

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CHAPTER 2

Simplex Method

2.1 INTRODUCTION

When there are more than two variables, graphical method cannot be used to solve optimization problems. As it was explained in the previous chapter that optimal solution exists always at the corner point at the feasible region, the simplex method is a systematic procedure at finding corner point solution and taking them for optimality. Simplex procedures are meant for profit maximization and if our objective is loss minimization then the problems has to be converted into profit maximization by multiplying the objective function by '–' sign before starting the simplex procedures.

While solving problems using simplex methods, we take slack variables, surplus variables and artificial variables. After solving the simplex problem, we can do the verification test of the result to make sure that the answer is correct. In many practical problems we want to find not only an optimal solution but also want to determine what happens to this optimal solution when certain changes are made in the system. For this we do the sensitivity. While doing sensitivity analysis if negative value appears in solution matrix, the matrix is not an optimum one and we move towards optimality by dual simplex procedure.

While solving problems using simplex methods if many artificial variables are to be used we will write a dual problem for the given problem and then solve it by simplex procedure. The intermediate steps and fundamentals involved in simplex procedure are given in the following problems.

Slack variables are those which are added to the constraint equations to get equal to sign ('=')

Example: 2.1

Assume that the following specify a generalized linear programming problem:

Maximize: $Z = x_1 - x_2 + 3x_3$

Subjected to $2x_1 - x_3 \le 2$

$x_1 + x_2 - x_3 \le 10$

$2x_1 - 2x_2 + 3x_3 \le 0$

$x_1, x_2, x_3 \ge 0$

Solve it by using Simplex Method.

Solution:

Let s_1, s_2, s_3 be slack variables then the constraints become

$$2x_1 - x_3 + s_1 = 2; \qquad x_1 + x_2 - x_3 + s_2 = 10$$

$$2x_1 - 2x_2 + 3x_3 + s_3 = 0; \qquad x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

$$\Rightarrow \qquad \text{Max } Z = x_1 - x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$$

Table 1:

- The coefficients of all the variables in the objective function are written in the first row against "C_j".
- The coefficients of the slack variables in the objective function are written in the first column under "Basis".
- The coefficients of all the variables in the constraints are written in the corresponding rows as shown below.
- The constants in the constraints are written under "b"
- The E_j values of the variables are the sum of the products of the coefficients in the respective columns and the corresponding basis.
- e.g., $2 x 0 + 1 x 0 + 2 x 0 = 0$

Basis \ C_j	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	2	0	– 1	1	0	0	2	
0 s_2	1	1	– 1	0	1	0	10	
0 s_3	2	– 2	3	0	0	1	0	
E_j	0	0	0	0	0	0		
$E_j - C_j$	–1	1	– 3	0	0	0		

- In $E_j - C_j$ row the value represents the corresponding profit on each machine.
- Select the maximum negative value in the pivot row, so that we select the max profit or min loss. The column corresponding to that value is called as entering column (C_i). This row is called pivot or key column.
- The θ values are calculated by dividing the values under b with corresponding elements in the entering column.
- Select the minimum positive value (including '0')in the $\left(\theta = \frac{b}{C_i}\right)$ column,
- This represents idleness of the machine as shown in above table. The corresponding row is called the leaving row. This row is called pivot or key row.

Basis \ C_j	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	2	0	– 1	1	0	0	2	– 2
0 s_2	1	1	– 1	0	1	0	10	– 10
0 s_3	2	– 2	$\boxed{3}$ *	0	0	1	0	0 ← Min +ve Value including zero (Idleness)
E_j	0	0	0	0	0	0		
$E_j - C_j$	–1	1	– 3	0	0	0		

Max –ve value [means 3 units of profit] $\frac{d^2x}{dt^2} = -\text{ve(max)}$

- The element common in the entering column and leaving row is called pivotal value.
- The basis variable in the leaving row is replaced by the variable in entering column. i.e., s_3 leaves the basis, x_3 enters the basis ($s_3 \Leftrightarrow x_3$).

Basis ↓ C_j →	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	2	0	– 1	1	0	0	2	– 2
0 s_2	1	1	– 1	0	1	0	10	– 10
0 s_3	2	– 2	$\boxed{3}$ *	0	0	1	0	0
E_j	0	0	0	0	0	0		
$E_j - C_j$	–1	1	– 3	0	0	0		

Table 2:

- Pivotal value in row is made '1' by applying only row operations and all other values in the column are made zero and only by following row operations.

$$R_{3\text{ new}} = R_{3\text{ old}} \div 3$$
$$R_{1\text{ new}} = R_{1\text{ old}} + R_{3\text{ new}}$$
$$R_{2\text{ new}} = R_{2\text{ old}} + R_{3\text{ new}}$$

- The E_j and $E_j - C_j$ values are calculated as explained earlier.

Basis ↓ C_j →	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$	
0 s_1	8/3	–2/3	0	1	0	1/3	2	– 3	
0 s_2	5/3	$\boxed{1/3}$ *	0	0	1	1/3	10	30	← Leaving row
3 x_3	2/3	– 2/3	1	0	0	1/3	0	0	
E_j	2	– 2	3	0	0	1			
$E_j - C_j$	1	– 1	0	0	0	1			
		↑ Entering column							

- The entering column is identified and the θ values are calculated, then the leaving row are identified as explained earlier.

Note: The pivotal element should not be a negative number. So instead of "0" the next minimum positive number (here "30") is taken for the leaving row.

Basis ↓ C_j →	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	8/3	–2/3	0	1	0	1/3	2	– 3
0 s_2	5/3	$\boxed{1/3}$ *	0	0	1	1/3	10	30
3 x_3	2/3	– 2/3	1	0	0	1/3	0	0
E_j	2	– 2	3	0	0	1		
$E_j - C_j$	1	– 1	0	0	0	1		

- For the new simplex table, the pivotal element is selected according to previous procedure.
- s_2 leaves the basis, x_2 enters the basis.

Table 3:

- Pivotal value in row is made '1' by applying only row operations and all other values in the column are made zero and only by following row operations.

$$R_{2\,\text{new}} = R_{2\,\text{old}} \div (1/3)$$
$$R_{1\,\text{new}} = R_{1\,\text{old}} + (2/3)\,R_{2\,\text{new}}$$
$$R_{3\,\text{new}} = R_{3\,\text{old}} + (2/3)\,R_{2\,\text{new}}$$

- The E_j and $E_j - C_j$ values are calculated as explained earlier.

Basis \ C_j	1 x_1	– 1 x_2	3 x_3	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	6	0	0	1	2	1	22	
0 x_2	5	1	0	0	3	1	30	
3 x_3	4	0	1	0	2	1	20	
E_j	7	– 1	3	0	3	2		
$E_j - C_j$	6	0	0	0	3	2		

- As all the elements of $E_j - C_j$ row are positive, it means it is an optimum matrix.
- The variable present in the basis are called basic variables.
- The variables which are not in the basis but are in the objective function are called non basic variables. (The non-basic variables are assigned zero solution.)
- So solution can be read it out from matrix i.e.,

$$\left.\begin{array}{l} s_1 = 22 \\ x_2 = 30 \\ x_3 = 20 \end{array}\right\} \text{Basic variables} \qquad \left.\begin{array}{l} s_2 = 0 \\ s_3 = 0 \\ x_1 = 0 \end{array}\right\} \text{Non-basic variables}$$

$\therefore$ by putting $(x_1, x_2, x_3) = (0, 30, 20)$ in the objective function the maximum value of

$Z = x_1 - x_2 + 3x_3 = 0 - 30 + 60 = 30$

Example: 2.2

Assume that the following specify a generalized linear programming problem:

Maximize: $Z = 45x + 40y$

Subjected to $2x + y \le 90$

$x + 2y \le 80$

$x + y \le 50$

$x, y \ge 0$

Solve it by using Simplex Method.

Solution:

Let s_1, s_2, s_3 be slack variables then the constraints become

$$\begin{aligned} 2x + y + s_1 &= 90 \\ x + 2y + s_2 &= 80 \\ x + y + s_3 &= 50 \\ x, y, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

$\Rightarrow$ $Z = 45x + 40y + 0s_1 + 0s_2 + 0s_3$

- The above information is tabulated.
- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.

Basis \ C_j	45 x	40 y	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	$\boxed{2}$ *	1	1	0	0	90	45 ←
0 s_2	1	2	0	1	0	80	80
0 s_3	1	1	0	0	1	50	50
E_j	0	0	0	0	0		
$E_j - C_j$	– 45 ↑	– 40	0	0	0		

Table-1

s_1 leaves the basis and x enters into the basis.

- The following row operations are performed:

$$R_{1\,\text{new}} = R_{1\,\text{old}}/2$$
$$R_{2\,\text{new}} = R_{2\,\text{old}} - R_{1\,\text{new}}$$
$$R_{3\,\text{new}} = R_{3\,\text{old}} - R_{1\,\text{new}}$$

- Then the table 1 becomes table 2.
- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- s_3 leaves the basis, and y enters the basis.

Basis \ C_j	45 x	40 y	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
45 x	1	½	1/2	0	0	45	90
0 s_2	0	3/2	–1/2	1	0	35	70/3
0 s_3	0	*1/2	–1/2	0	1	5	10 ←
E_j	45	45/2	45/2	0	0		
$E_j - C_j$	0	–35/2 ↑	45/2	0	0		

Table-2

- The following row operations are performed:

$$R_{1\,\text{new}} = R_{1\,\text{old}}\,(R_{3\,\text{new}})/2$$
$$R_{2\,\text{new}} = R_{2\,\text{old}} - 3\,(R_{3\,\text{new}})/2$$
$$R_{3\,\text{new}} = 2R_{3\,\text{old}}$$

- Then the table 2 becomes

Basis \ C_j	45 x	40 y	0 s_1	0 s_2	0 s_3	b
45 x	1	0	1	0	– 1	40
0 s_2	0	0	1	1	– 3	20
40 y	0	1	– 1	0	2	10
E_j	45	40	5	0	35	
$E_j - C_j$	0	0	5	0	35	

Table-3

- E_j and $E_j - C_j$ are calculated.
- '$E_j - C_j$' Values present in the matrix are positive. So it is an optimum matrix.
- So solution can be read it out from matrix i.e.,

$$\left.\begin{aligned} x &= 40 \\ y &= 10 \\ s_2 &= 20 \end{aligned}\right\} \text{Basic variables} \qquad \left.\begin{aligned} s_1 &= 0 \\ s_3 &= 0 \end{aligned}\right\} \text{Non-basic variables}$$

$\therefore$ by putting $(x, y) = (40, 10)$ in the objective function the maximum value of $Z = 45x + 40y = 2200$

- **Surplus variables** are those which are subtracted from the constraint equation to get equal to sign ('=')
- **Artificial variables** are those which are added to the constraint equations whenever we introduce a surplus variable in order to get the initial feasible solution, which incurs a heavy loss in the above function. And so we assign a large penalty "$-M$" to these variables in the objective function.

Example: 2.3

Assume that the following specify a generalized linear programming problem:

Minimize: $Z = 4x_1 + x_2$

Subjected to $3x_1 + 4x_2 \geq 20$

$x_1 + 5x_2 \leq 15$

$x_1, x_2 \geq 0$

Solve it by using Big *M* Method or artificial variable Method.

Solution:

By introducing surplus variable s_1 artificial variable a_1 and slack variable s_2 the problem may be written as follows:

Maximize: $Z = -4x_1 - x_2 + 0s_1 - Ma_1 + 0s_2$

Subject to: $3x_1 + 4x_2 - s_1 + a_1 = 20$

$x_1 + 5x_2 + s_2 = 15$

$x_1, x_2, s_1, s_2, a_1 \geq 0$

- The above information is tabulated.
- In any constraint if artificial variable is present that should enter first in the basis. (Once the artificial variable leaves the basis it will never enter again, so from the next simplex table onwards that variable can be deleted permanently.)
- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.

Basis \ C_j	-4 x_1	-1 x_2	0 s_1	0 s_2	$-M$ a_1	b	$\theta = \frac{b}{C_i}$
$-Ma_1$	3	4	-1	0	1	20	5
$0\ s_2$	1	$\boxed{5}^*$	0	1	0	15	3
E_j	$-3M$	$-4M$	M	0	$-M$		
$E_j - C_j$	$-3M+4$	$-4M+1$	M	0	0		

Table-1

- S_2 leaves the basis and X_2 enters the basis.
- The following row operations are performed:

$$R_{2\,\text{new}} = (R_{2\,\text{old}})/5; \qquad R_{1\,\text{new}} = R_{1\,\text{old}} - 4\,(R_{2\,\text{new}})$$

- Then the table 1 becomes

Basis \ C_j	-4 x_1	-1 x_2	0 s_1	0 s_2	$-M$ a_1	b	$\theta = \frac{b}{C_i}$
$-Ma_1$	$\boxed{11/5}$*	0	-1	$-4/5$	1	8	40/11 ←
$-1\ x_2$	1/5	1	0	1/5	0	3	15
E_j	$-\frac{11M-1}{5}$	-1	M	$\frac{4M-1}{5}$	M		
$E_j - C_j$	$\frac{-11M+19}{5}$ ↑	0	M	$\frac{4M-1}{5}$	0		

Table-2

- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- a_1 leaves the basis and x_1 enters the basis.
- As a_1 leaves the basis, it will never come back to the basis again, so that particular column is removed from matrix.
- The following row operations are performed:

$$R_{1\,\text{new}} = 5\,(R_{1\,\text{old}})/11; \qquad R_{2\,\text{new}} = R_{2\,\text{old}} - (R_{1\,\text{new}})/5$$

- Then the table 2 becomes table 3.
- E_j and $E_j - C_j$ are calculated.
- It is an optimum matrix since all the values obtained in '$E_j - C_j$' row are positive

Basis \ C_j	-4 x_1	-1 x_2	0 s_1	0 s_2	b
$-4\ x_1$	1	0	$-5/11$	$-4/11$	40/11
$-1\ x_2$	0	1	1/11	3/11	25/11
E_j	-4	-1	19/11	13/11	
$E_j - C_j$	0	0	19/11	13/11	

Table-3

- So solution can be read it out from matrix i.e.,

$$\left.\begin{aligned} x_1 &= \frac{40}{11} \\ x_2 &= \frac{25}{11} \end{aligned}\right\} \text{Basic variables} \qquad \left.\begin{aligned} s_1 &= 0 \\ s_2 &= 0 \\ a_1 &= 0 \end{aligned}\right\} \text{Non-basic variables}$$

$\therefore$ by putting $(x_1, x_2) = \left(\frac{40}{11}, \frac{25}{11}\right)$ in the objective function the maximum value of

$$Z = -4x_1 - x_2 = \frac{185}{11}$$

Example: 2.4

Assume that the following specify a generalized linear programming problem:

Minimum: $Z = 3x_1 + 2x_2$

Subject to: $2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_2, x_2 \geq 0,$

Solve it by using Big M Method.

Solution:

By introducing slack variable s_1 surplusvariable s_2 and artificial variable a_2 the problem may be written as follows:

Maximize: $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 - Ma_2$

Subject to:

$$2x_1 + x_2 + s_1 = 2$$
$$3x_1 + 4x_2 - s_2 + a_2 = 12$$
$$x_1, x_2, s_1, s_2, a_2 \geq 0$$

- The above information is tabulated.
- In any constraint if artificial variable is present that should enter first in the basis. (Once the artificial variable leaves the basis it will never enter again, so from the next simplex table onwards that variable can be deleted permanently.)

Basis $\backslash$ C_j	-3 x_1	-2 x_2	0 s_1	0 s_2	$-M$ a_2	b	$\theta = \frac{b}{C_i}$
$0\ s_1$	2	$\boxed{1}$*	1	0	0	2	2 ←
$-Ma_2$	3	4	0	-1	1	12	3
E_j	$-3M$	$-4M$	0	M	$-M$		
$E_j - C_j$	$-3M + 3$	$-4M + 2$	0	M	0		

Table-1

- E_j and $E_j - C_j$ are calculated.
- The entering column is identified andthe θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- s_1 leaves the basis, x_2 enters the basis.
- The following row operations are performed

$$R_{2\text{ new}} \rightarrow R_{2\text{ old}} - 4R_{1\text{ new}}$$

- Then the table 1 becomes

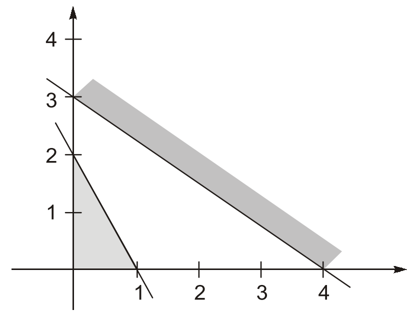
Basis $\backslash$ C_j	-3 x_1	-2 x_2	0 s_1	0 s_2	$-M$ a_1	b
$-2x_2$	2	1	1	0	0	2
$-Ma_2$	-5	0	-4	-1	1	4
E_j	$5M-4$	-2	$4M-2$	M	$-M$	
$E_j - C_j$	$5M-1$	0	$4M-2$	M	0	

Table-2

- It is an optimum matrix since all the values obtained in '$E_j - C_j$' row are positive.
- But the optimum matrix contains artificial variable, this indicates infeasible solution i.e., (there is no feasible region).

- If we do the same problem by graphical method we get

$$\frac{x_1}{1} + \frac{x_2}{2} \le 1 \qquad \frac{x_1}{4} + \frac{x_2}{3} \ge 1$$



Note: After selecting the entering column, if all the values appear to be negative, it indicates that the solution is unbounded.

Example: 2.5

Assume that the following specify a generalized linear programming problem:

Maximize: $Z = 6x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \le 30$

$3x_1 + 2x_2 \le 24$

$x_1 + x_2 \ge 3$

$x_1, x_2 \ge 0$

Solve it by using Big *M* Method.

Solution:

By introducing slack variables s_1, s_2 surplus variable s_3 and artificial variable a_1 the problem may be written as follows:

Maximize: $Z = 6x_1 + 4x_2 + 0.s_1 + 0.s_2 + 0.s_3 - Ma_1$

Subject to
$$2x_1 + 3x_2 + s_1 = 30$$
$$3x_1 + 2x_2 + s_2 = 24$$
$$x_1 + x_2 - s_3 + a_1 = 3$$
$$x_1, x_2, s_1, s_2, s_3, a_1 \ge 0$$

- The above information is tabulated.
- In any constraint if artificial variable is present that should enter first in the basis.
- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table

Basis \ C_j	6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	– M a_1	b	$\theta = \frac{b}{C_i}$
0 s_1	2	3	1	0	0	0	30	15
0 s_2	3	2	0	1	0	0	24	8
M a_1	$\boxed{1}$*	1	0	0	– 1	1	3	3 ←
E_j	– M	– M	0	0	M	– M		
$E_j - C_j$	– M – 6 ↑	– M – 4	0	0	M	0		

Table-1

- s_1 leaves the basis, x_2 enters the basis.
- The following row operations are performed:

$$R_{1\text{ new}} = R_{1\text{ old}} - 2R_{3\text{ new}}$$
$$R_{2\text{ new}} = R_{2\text{ old}} - 3R_{3\text{ new}}$$

- Then the table 1 becomes.
- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- s_2 leaves the basis, s_3 enters the basis.

Basis \ C_j	6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	0	1	1	0	2	24	12
0 s_2	0	– 1	0	1	$\boxed{3}$ *	15	5
6 x_1	1	1	0	0	– 1	3	– 3
E_j	6	6	0	0	– 6		
$E_j - C_j$	0	2	0	0	– 6		

Table-2

- The following row operations are performed:

$$R_{2\text{ new}} = (R_{2\text{old}})/3$$
$$R_{2\text{ new}} = R_{2\text{ old}} - 2R_{2\text{ new}}$$
$$R_{3\text{ new}} = R_{3\text{ old}} + R_{2\text{ new}}$$

- Then the table 2 becomes.

Basis \ C_j	6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	0	$\boxed{5/3}$ *	1	–2/3	0	14	42/5
0 s_3	0	–1/3	0	1/3	1	5	–15
6 x_1	1	2/3	0	1/3	0	8	12
E_j	6	4	0	2	0		
$E_j - C_j$	0	(0)	0	2	0		

Table-3

- E_j and $E_j - C_j$ are calculated.
- It is an optimum matrix since all the values obtained in '$E_j - C_j$' row are positive.
- So solution can be read it out from matrix i.e.,

$$\begin{array}{ll} s_1 = 14 & x_2 = 0 \\ s_3 = 5 & s_2 = 0 \\ x_1 = 8 & a_1 = 0 \end{array} \Rightarrow X_1 = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 14 \\ 0 \\ 5 \end{pmatrix}$$

- **Normally in simplex, zero will appear in $E_j - C_j$ row corresponding to every basic variable. If it so happens that zero appears below a non basic variables it means there are alternate solutions.**
- Here x_2 is a non basic variable and is having '0' in row. So, this is the case of alternate solution.
- To get the alternative solution the entering column is identified as x_2 is having 0' in and is not in the basis and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- The following row operations are performed:

$$R_{1\text{ new}} = 3(R_{1\text{old}})/5$$
$$R_{2\text{ new}} = R_{2\text{ old}} + (R_{1\text{ new}})/3$$
$$R_{3\text{ new}} = R_{3\text{ old}} - (R_{1\text{ new}})/3$$

- Then the table 3 becomes.

Basis \ C_j	6 x_1	4 x_2	0 s_1	0 s_2	0 s_3	b
4 x_2	0	1	3/5	–2/5	0	4275
0 s_3	0	0	1/5	1/5	1	39/5
6 x_1	1	0	–2/5	3/5	0	12/5
E_j	6	4	0	2	0	
$E_j - C_j$	0	$\boxed{0}$	0	2	0	

Table-4

- E_j and $E_j - C_j$ are calculated.
- It is an alternative optimum matrix since all the values obtained in '$E_j - C_j$' row are positive.
- So solution can be read it out from matrix i.e.,

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 45/2 \\ 0 \\ 0 \\ 39/5 \end{pmatrix}$$

- Here alternate solution means infinitely many solutions may be there because the objective function line is coinciding with the constraint line at the farthest point. The infinite number of solutions is represented as

$$x = (\lambda)x_2 + (1 - \lambda)x_2$$
$$y \in (0, 1)$$

Example: 2.6

Assume that the following specify a generalized linear programming problem:

Maximum: $Z = 2x_1 + x_2$

Subject to: $4x_1 + 3x_2 \le 12$

$4x_1 + x_2 \le 8$

$4x_1 - x_2 \le 8$

$x_1, x_2 \ge 0$

Solve it by using Simplex Method.

Solution:

By introducing slack variables s_1, s_2 and s_3 the problem may be written as follows:

Maximum: $Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$

Subjected to:
$$4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8,$$
$$4x_1 - x_2 + s_3 = 8,$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

The above information is tabulated.

Basis \ C_j	2 x_1	1 x_2	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	4	3	1	0	0	12	3
0 s_2	4	1	0	1	1	8	2
0 s_3 ←	$\boxed{4}$*	– 1	0	0	0	8	2 ←
E_j	0	0	0	0	0		
$E_j - C_j$	– 2 ↑	– 1	0	0	0		

Table-1

- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated.
- **In the 'θ'column when two minimum positive values are same and we are not able to decide which row to enter it is called degeneracy in simplex.**
- To find degenerate solution of simplex we have to find minimum $\left(\frac{s_i}{C_i}\right)$.

  Here in s_1 column both $\left(\frac{s_i}{C_i}\right)$ are same. But in s_2 column Min $\left(\frac{1}{4}, \frac{0}{4}\right)$ is '0'. So s_3 is identified as leaving row.
- The pivotal element/value is identified from the table.
- s_3 leaves the basis, x_1 enters the basis.
- The following row operations are performed:
$$R_{3\,\text{new}} = (R_{3\,\text{old}})/4$$
$$R_{1\,\text{new}} = R_{1\,\text{old}} - 4(R_{3\,\text{new}})$$
$$R_{2\,\text{new}} = R_{2\,\text{old}} - 4(R_{3\,\text{new}})$$
- Then the table 1 becomes.

Basis \ C_j	2 x_1	1 x_2	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	0	4	1	0	– 1	4	1
0 s_2 ←	0	$\boxed{2}$*	0	1	– 1	0	0 ←
2 x_1	1	–1/4	0	0	1/4	2	– 8
E_j	2	–1/2	0	0	1/2	4	– 16
$E_j - C_j$	0	–3/2 ↑	0	0	1/2		

Table-2

- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- s_2 leaves the basis, x_2 enters the basis.
- The following row operations are performed:

$$R_{2\,new} = (R_{2\,old})/2$$
$$R_{1\,new} = R_{1\,old} - 4(R_{2\,new})$$
$$R_{3\,new} = R_{3old} + (R_{2\,new})/4$$

- Then the table 1 becomes.

Basis \ C_j	2 x_1	1 x_2	0 s_1	0 s_2	0 s_3	b	$\theta = \frac{b}{C_i}$
0 s_1	0	0	1	– 2	$\boxed{1}$ *	4	4
1 x_2	0	1	0	1/2	–1/2	0	0
2 x_1	1	0	0	1/8	1/8	2	16
E_j	2	1	0	3/4	–1/4		
$E_j - C_j$	0	0	0	3/4	–1/4		

Table-3

- E_j and $E_j - C_j$ are calculated.
- The entering column is identified and the θ values are calculated, then the leaving row are identified.
- The pivotal element/value is identified from the table.
- s_1 leaves the basis, s_3 enters the basis.
- The following row operations are performed:

$$R_{2\,new} = R_{2\,old} + (R_{1\,new})/2$$
$$R_{3\,new} = R_{3\,old} - (R_{1\,new})/8$$

- Then the table 3 becomes.

Basis \ C_j	2 x_1	1 x_2	0 s_1	0 s_2	0 s_3	b
0 s_3	0	0	1	– 2	1	4
1 x_2	0	1	1/2	–1/2	0	2
2 x_1	1	0	–1/8	3/8	0	3/2
E_j	2	1	1/4	1/4	0	
$E_j - C_j$	0	0	1/4	1/4	0	

Table-4

- E_j and $E_j - C_j$ are calculated.
- It is an optimum matrix since all the values obtained in '$E_j - C_j$' row are positive.

- So solution can be read it out from matrix i.e.,

$$\begin{Bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{Bmatrix} = \begin{Bmatrix} 3/2 \\ 2 \\ 0 \\ 0 \\ 4 \end{Bmatrix}$$

- By putting $(x_1, x_2) = \left(\frac{3}{2}, 2\right)$ in the objective function the maximum value of $Z = 2x_1 + x_2 = 5$

2.2 VERIFICATION OF SIMPLEX RESULTS

If the product of the matrix under the slack or surplus variables in the final optimal table and the matrix of constants in the given constraints is equal to the solution matrix then the solution is correct.

Example 2.7

Assume that the following specify a generalized linear programming problem:

Maximum: $Z = 3x_1 + 2x_2$

Subject to: $x_1 + x_2 \le 6$

$2x_1 + x_2 \le 8$

$-x_1 + x_2 \le 1,\ x_2 \le 2$

Solve it by using Simplex Method and verify the result.

Solution:

By introducing slack variable s_1, s_2, s_3 and s_4 the problem may be written as follows:

Maximum: $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

$$\begin{aligned} x_1 + x_2 + s_1 &= 6 \\ 2x_1 + x_2 + s_2 &= 8 \\ -x_1 + x_2 + s_3 &= 1 \\ x_2 + s_4 &= 2 \\ x_1, x_2, s_1, s_2, s_3, s_4 &\ge 0 \end{aligned}$$

- The above information is tabulated.

Basis / C_j	3 x_1	2 x_2	0 s_1	0 s_2	0 s_3	0 s_4	b	$\theta = \frac{b}{C_i}$
0 s_1	1	1	1	0	0	0	6	6
0 s_2	$\boxed{2}$	1	0	1	0	0	8	4
0 s_3	– 1	1	0	0	1	0	1	– 1
0 s_4	0	1	0	0	0	1	2	∞
E_j	0	0	0	0	0	0		
$E_j - C_j$	– 3	– 2	0	0	0	0		

Table-1

PRACTICE QUESTIONS

1. Which one of the following subroutines does a computer implementations linear programming by the Simplex method use?
(a) Finding a root of a polynomial
(b) Finding the determinant of a matrix
(c) Finding the Eigen values of a matrix
(d) Solving a system of linear equations

2. A variable which has no physical meaning, but is used to obtain an initial basic feasible solution to the linear programming problem is referred to as:
(a) Basic variable
(b) Non-basic variable
(c) Artificial variable
(d) Basis

3. In the solution of a linear programming problem by simplex method, if during iteration, all ratios of right-hand side b_1 to the coefficients of entering variable a are found to be negative, it implies that the problem has:
(a) Infinite number of solutions
(b) Infeasible solution
(c) Degeneracy
(d) unbound solutions

4. Consider the following statements regarding the characteristics of the standard form of a linear programming problem:
   1. All the constraints are expressed in the form of equations.
   2. The right-hand .side of each constraint equation is non-negative.
   3. All the decision variables are non-negative.

   Which of these statements are correct?
(a) 1, 2 and 3
(b) 1 and 2
(c) 2 and 3
(d) 1 and 3

5. Consider the following statements:
   1. A linear programming problem with three variables and two constraints can be solved by graphical method.
   2. For solutions of a linear programming problem with mixed constraints, Big-M method can be employed.
   3. In the solution process of a linear programming problem using Big-M method, when an artificial variable leaves the basis, the column of the artificial variable can be removed from all subsequent tables.

   Which of these statements are correct?
(a) 1, 2 and 3
(b) 1 and 2
(c) 1 and 3
(d) 2 and 3

6. Consider the following statements regarding linear programming:
   1. Dual of a dual is the primal.
   2. When two minimum ratios of the right-hand side to the coefficient in the key column are equal, degeneracy may take place.
   3. When an artificial variable leaves the basis, its column can be deleted from the subsequent Simplex tables.

   Select the correct answer from the codes given below:
(a) 1, 2 and 3
(b) 1 and 2
(c) 2 and 3
(d) 1 and 3

7. The primal of a LP problem is maximization of objective function with 6 variables and 2 constraints. Which of the following correspond to the dual of the problem stated?
   1. It has 2 variables and 6 constraints.
   2. It has 6 variables and 2 constraints.
   3. Maximization of objective function.
   4. Minimization of objective function.

   Select the correct answer using the codes given below:
   (a) 1 and 3 (b) 1 and 4
   (c) 2 and 3 (d) 2 and 4

8. In the solution of linear programming problems by Simplex method, for deciding the leaving variable:
   (a) The maximum negative coefficient in the objective function row is selected.
   (b) The minimum positive ratio of the right-hand side to the first decision variable is selected.
   (c) The maximum positive ratio of the right-hand side to the coefficient in the key column is selected.
   (d) The minimum positive ratio of the right-hand side to the coefficient in the key column is selected.

9. A tie for leaving variables in simplex procedure implies:
   (a) Optimality (b) Cycling
   (c) No solution (d) Degeneracy

10. In a linear programming problem, if a basic solution has no more than m positive x_j(j = 1, 2......n), it is called:
    (a) Basic feasible solution
    (b) Unbounded solution
    (c) Non-degenerate basic feasible solution
    (d) None of the above

11. Which one of the following is the correct statement?
    In the standard form of a linear programming problem, all constraints are
    (a) of less than or equal to, type
    (b) of greater than or equal to, type
    (c) in the form of equations
    (d) some constraints are of less than equal to, type and some of greater than equal to, type

12. Which one of the following statements is not correct?
    (a) A linear programming problem with 2 variables and 3 constraints can be solved by Graphical method.
    (b) In big-M method if the artificial variable cannot be driven out it depicts an optimal solution.
    (c) Dual of dual is the primal problem.
    (d) For mixed constraints either big-M method or two phase method can be employed.

13. A linear programming problem with mixed constraints (some constraints of ≤ type and some of ≥ type) can be solved by which of the following methods?
    (a) Big-M method (b) Hungarian method
    (c) Branch and bound technique (d) Least cost method

24. Which one of the following is true in case of simplex method oflinear programming?
   (a) The constants of constraints equation may be positive or negative.
   (b) Inequalities are not converted into equations.
   (c) It cannot be used for two-variable problems.
   (d) The simplex algorithm is an iterative procedure.

25. Match **List-I (Persons with whom the models are associated)** with **List-II (Models)** and select the correct answer:

List-I	List-II
A. J. Von Newmann	**1.** Waiting lines
B. G. Dantzig	**2.** Simulation
C. A.K. Erlang	**3.** Dynamic programming
D. Richard Bellman	**4.** Competitive strategies
	5. Allocation by simplex method

Codes:

	A	B	C	D
(a)	2	1	5	4
(b)	4	5	1	3
(c)	2	5	1	4
(d)	4	1	5	3

❍❍❍❍

ANSWERS

SIMPLEX METHOD

1. (d)	**2.** (c)	**3.** (d)	**4.** (a)	**5.** (d)	**6.** (a)	**7.** (b)
8. (d)	**9.** (d)	**10.** (a)	**11.** (d)	**12.** (b)	**13.** (a)	**14.** (a)
15. (d)	**16.** (d)	**17.** (a)	**18.** (b)	**19.** (d)	**20.** (b)	**21.** (a)
22. (d)	**23.** (b)	**24.** (d)	**25.** (b)			